

Sieves for Twin Primes in Class I

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Abstract

Sieves are constructed for twin primes in class I, which are of the form $2m \pm D$, $D \geq 3$ odd. They are characterized by their twin-D-I rank m . They have no parity problem. Non-rank numbers are identified and counted using odd primes $p \geq 5$. Twin-D-I ranks and non-ranks make up the set of positive integers. Regularities of non-ranks allow obtaining the number of twin-D-I ranks. It involves considerable cancellations so that the asymptotic form of its main term collapses to the expected form, but its coefficient depends on D .

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1 Introduction

Sieve theory has developed over almost a century into a versatile tool of number theory [1], [2],[3],[4]. For twin primes it is the method of choice. The first genuine pair sieve constructed in Ref. [5] for ordinary twin primes is adapted to twin primes at distance $2D \geq 6$ with D odd and fixed throughout, except for examples. Their arithmetic is fairly different from distance 2 (or

4) [5],[6], because the half-distance D has at least one odd prime divisor, whereas for ordinary prime twins of the form $6m \pm 1$ it has none.

Prime numbers $p \geq 5$ are well known to be of the form [7] $6m \pm 1$. Since 2, 3 are not of the form $6m \pm 1$, they are excluded as primes in the following. An ordinary twin prime occurs when both $6m \pm 1$ are prime. Twin primes at distance $2D$ can be written similarly as $2m \pm D$, D odd being in class I of the classification [8], [9] of all twin primes, the same class as ordinary twins of the form $2(3m) \pm 1$.

Definition 1.1. The base set of the sieve consists of all positive integers; it is partitioned into twin-D-I ranks and non-ranks. A number m is called *twin-D-I rank* if $2m \pm D$ are both prime. If $2m \pm D$ are not both prime, then m is a *non-rank*. Multiples nq of divisors $q \mid D$ are trivial non-ranks because $2nq \pm D$ are never prime.

Example 1.2. Twin-D-I ranks for $D = 3$ are 4, 5, 7, 8, 10, \dots ; for $D = 5$ they are 3, 4, 6, 9, 11, 12, \dots . Non-ranks for $D = 3$ are 6, 9, 11, 12, 13, \dots ; for $D = 5$ they are 5, 7, 8, 10, 13, 14, \dots .

Only non-ranks have sufficient regularity and abundance allowing us to determine the number of twin-D-I ranks. Therefore, our main focus is on non-ranks, their symmetries and abundance.

In Sect. 2 the twin-D-I prime sieve is constructed based on non-ranks. In Sect. 3 non-ranks are identified in terms of their main properties and then, in Sect. 4, they are counted. In Sect. 5 twin-D-I ranks are isolated and counted. Conclusions are summarized and discussed in Sect. 6.

2 Twin Ranks, Non-Ranks and Sieve

It is our goal here to construct twin-D-I prime sieves in detail. We need the following arithmetical function [5],[6].

Definition 2.1. Let x be real. Then $N(x)$ is the integer nearest to x . The ambiguity for $x = n + \frac{1}{2}$ with integral n will not arise in the following.

Lemma 2.2. Let $p \geq 5$ be prime. Then

$$N\left(\frac{p}{6}\right) = \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}; \\ \frac{p+1}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases} \quad (1)$$

Proof. This is obvious from Def. 2.1 by substituting $p = 6m \pm 1$. \diamond

Lemma 2.3 Let $p \geq 5$ be prime and $(p, D) = 1$. Then the numbers

$$k(n, p)^+ = np + 3DN\left(\frac{p}{6}\right), \quad n = 0, 1, 2, \dots$$

$$k(n, p)^- = np - 3DN\left(\frac{p}{6}\right) > 0, \quad n > \frac{D+1}{2} \quad (2)$$

are non-ranks. There are $2 = 2^{\nu(p)}$ (single) non-rank progressions to the prime p .

(a) If $p \equiv 1 \pmod{6}$ the non-rank $k(n, p)^+$ generates the pair

$$2k(n, p)^+ \pm D = ((2n + D)p - 2D, (2n + D)p), \quad (3)$$

and the non-rank $k(n, p)^-$ the pair

$$2k(n, p)^- \pm D = ((2n - D)p, (2n - D)p + 2D), \quad 2n > D + 1. \quad (4)$$

(b) If $p \equiv -1 \pmod{6}$ the non-rank $k(n, p)^+$ generates the pair

$$2k(n, p)^+ \pm D = ((2n + D)p, (2n + D)p + 2D); \quad (5)$$

and the non-rank $k(n, p)^-$ the pair

$$2k(n, p)^- \pm D = ((2n - D)p - 2D, (2n - D)p), \quad 2n > D + 1. \quad (6)$$

All pairs contain a composite number.

Clearly, all these non-ranks are symmetrically distributed at equal distances $3DN(p/6)$ from multiples of each prime $p \geq 5$, except for prime divisors of D .

Proof. Let $p \equiv 1 \pmod{6}$ be prime and $n \geq 0$ an integer. Then $2k(n, p)^+ \pm D = 2np + D(p - 1) \pm D$ by Lemma 2.2 and $2k^+$ is sandwiched by the pair in Eq. (3) which contains a composite number. Hence $k(n, p)^+$ is a non-rank. For $2n > D + 1$, the same happens in Eq. (4), so k^- is a non-rank.

If $p \equiv -1 \pmod{6}$ and prime, then $2k(n, p)^+ \pm D = 2np + D(p + 1) \pm D$ by Lemma 2.2 and k^+ leads to the pair in Eq. (5) which contains a composite number again. For $2n > D + 1$, the same happens in Eq. (6), so k^- is a non-rank. \diamond

The $k(n, p)^\pm$ yield pairs $2k^\pm \pm D$ with one or two composite entries that are twin-D-I prime analogs of multiples np , $n > 1$, of a prime p in Eratosthenes' prime sieve [7]. Non-ranks form the sieving set.

The converse of Lemma 2.3 holds, i.e. nontrivial non-ranks are organized in terms of arithmetic progressions with primes ≥ 5 (and their products) as periods. This makes it a cornerstone of the pair sieves.

Lemma 2.4. *If k is a nontrivial non-rank, there is a prime $p \geq 5$ and an integer λ so that $k = k(\lambda, p)^+$ or $k = k(\lambda, p)^-$.*

Proof. If $k \equiv 0 \pmod{3}$ and $D \equiv 1 \pmod{6}$, then $2k + D \equiv 1 \pmod{6}$. Let $2k + D = pK$ be composite, where $p \geq 5$ is the smallest prime divisor $\nmid D$. Then $2k + D \neq 3^\nu$, $\nu \geq 1$ obviously. If $p \equiv 1 \pmod{6}$ then $K \equiv 1 \pmod{6}$. So $p = 6m + 1$, $K = 6\kappa + 1$, $k = 3k'$, $D = 6d + 1$ and

$$2k + D = 6^2 m\kappa + 6(m + \kappa) + 1, \quad k' + d = 6m\kappa + m + \kappa = p\kappa + \frac{p-1}{6}. \quad (7)$$

Hence

$$k = 3p\kappa - 3d + 3\frac{p-1}{6} = p(3\kappa - \frac{D-1}{2}) + 3DN(\frac{p}{6}) \quad (8)$$

and $\lambda = 3\kappa - \frac{D-1}{2}$.

If $p \equiv -1 \pmod{6}$ then $K \equiv -1 \pmod{6}$, i.e. $p = 6m - 1$, $K = 6\kappa - 1$ and

$$2k + D = 6^2 m\kappa - 6(m + \kappa) + 1, \quad k' + d = 6m\kappa - (m + \kappa) = p\kappa - \frac{p+1}{6}, \quad (9)$$

then

$$k = 3p\kappa - 3d - 3\frac{p+1}{6} = p(3\kappa + \frac{D-1}{2}) - 3DN(\frac{p}{6}), \quad (10)$$

and $\lambda = 3\kappa + \frac{D-1}{2}$.

If $2k - D = pK$ i.e. is composite and $p = 6m + 1$, then $K = 6\kappa - 1$ because $2k - D = 6k' - 6d - 1$. Hence

$$2k - D = 6^2 m\kappa + 6(\kappa - m) - 1, \quad k' - d = p\kappa - \frac{p-1}{6} \quad (11)$$

and

$$k = p(3\kappa + \frac{D-1}{2}) - 3DN(\frac{p}{6}). \quad (12)$$

So $\lambda = 3\kappa + \frac{D-1}{2}$.

If $p = 6m - 1$ then $K = 6\kappa + 1$ and $2k - D = 6(k' - \kappa) - 1$,

$$2k - D = 6^2 m\kappa + 6(m + \kappa) - 1, \quad k' + d = 6m\kappa - (m + \kappa) = p\kappa - \frac{p+1}{6}. \quad (13)$$

Hence

$$k = 3p\kappa + 3d - 3\frac{p+1}{6} = p(3\kappa - \frac{D-1}{2}) + 3DN(\frac{p}{6}) \quad (14)$$

and $\lambda = 3\kappa - \frac{D-1}{2}$.

If $D \equiv 3 \pmod{6}$, then $2k \pm D = 3(2k' + 2d + 1)$ is always composite. These trivial non-rank cases are ignored in the following, except when counting non-ranks.

If $D = 6d - 1 \equiv -1 \pmod{6}$, then there are four cases for $2k + D$ or $2k - D$ composite, combined with the options for p and K as in (i), which are all handled the same way and then lead to similar results.

(ii) If $k = 3k' + 1 \equiv 1 \pmod{3}$ and $D = 6d - 1 \equiv -1 \pmod{6}$, then $2k + D = 6k' + 6d + 1 \equiv 1 \pmod{6}$. Let $2k + D = pK$ be composite and $p = 6m + 1$. Then $K = 6\kappa + 1$ and

$$2k + D = p \cdot K = (6m + 1)(6\kappa + 1) = 6^2 m\kappa + 6(m + \kappa) + 1. \quad (15)$$

Hence

$$\begin{aligned} k' + d &= 6m\kappa + m + \kappa = p\kappa + \frac{p-1}{6}, \\ k &= 3k' + 1 = [3\kappa - \frac{D+1}{2}]p + 3DN(\frac{p}{6}), \end{aligned} \quad (16)$$

so $\lambda = 3\kappa - \frac{D+1}{2}$.

If $D = 6d + 3$, then $2k + D = 6k' + 5 + 6d \equiv -1 \pmod{6} = pK$. If $p = 6m + 1$, then $K = 6\kappa - 1$ and

$$\begin{aligned} 2k + D &= 6^2 m\kappa + 6(\kappa - m) - 1, \\ k' + d + 1 &= p\kappa - \frac{p-1}{6}, \\ k &= 3p\kappa - 3(d + 1) + 1 - \frac{p-1}{2} \\ &= p(3\kappa - \frac{D-3}{2} - 2) + 3DN(\frac{p}{6}), \end{aligned} \quad (17)$$

so $\lambda = 3\kappa - \frac{D-3}{2} - 2$. The case where $p = 6m - 1$, $K = 6\kappa + 1$ is handled similarly. All other cases lead to trivial non-ranks.

(iii) The cases for $k = 3k' - 1$ are similar and handled in the same way. \diamond

Theorem 2.5. (Prime Pair Sieves) *Let $\mathcal{P} = \{(2m - D \geq 3, 2m + D) : 2m \not\equiv 0 \pmod{q}, q \mid D, m \text{ integral}\}$ be the set of pairs with entries ≥ 3 of natural numbers at distance $2D$, D odd. Upon striking all pairs identified by non-ranks of Lemma 2.3, only (and all) twin- D -I prime pairs are left.*

Since after sieving only twin- D -I ranks are left that lead to prime pairs at distance $2D$ (and no composites) the sieves have no parity problem.

Proof. For $2m - D \geq 3$ divide $2m \pm D$ by all primes $p < \sqrt{2m + D}$. Then m is a non-rank if there is a prime p such that $(2m - D)/p$ or $(2m + D)/p$ (or both) is integral. All such m are struck from the set of positive integers. Then all remaining integers are twin-D-I ranks. \diamond

3 Identifying Non-Ranks

Here it is our goal to characterize and systematically identify non-ranks among natural numbers.

Definition 3.1 Let $p \geq 5$ be the minimal prime of a non-rank. Then p is its parent prime.

The non-ranks to parent prime 5 are, by Lemma 2.3,

$$k^+ = 5n + 3D, n > -[\frac{3D}{5}]; k^- = 5n - 3D, 5n > 3D; (n, D) = 1, 5 \nmid D, \quad (18)$$

where $[x]$ is the largest integer below x , as usual. These k^\pm form the set $\mathcal{A}_5^- = \mathcal{A}_5$.

Note that 5 is the most effective non-rank generating prime number (except when $5|D$). If it were excluded like 3 then many numbers would be missed as non-ranks.

In contrast to ordinary twin primes [5] the arithmetic function values $N(p'/6), N(p/6)$ do not suffice to characterize twin-D-I primes $p' = p + 2D$.

Lemma 3.2. Let $p' > p$ be primes. Then $p' = p + 2D$ are ordinary prime twins iff $N(\frac{p'}{6}) = N(\frac{p}{6})$.

Proof. See Theor. 3.6 of Ref. [5].

Lemma 3.2 generalizes to $D \geq 3$ as follows.

Corollary 3.3. Let $p' > p \geq 5$ be primes with $p' \equiv p \pmod{6}$.

(i) If $D \equiv 0 \pmod{3}$ then $p' = p + 2D$ holds iff $N(\frac{p'}{6}) = \frac{D}{3} + N(\frac{p}{6})$.

(ii) If $D \equiv 1 \pmod{3}$ then $p' = p + 2(D - 1)$ iff $N(\frac{p'}{6}) = \frac{D-1}{3} + N(\frac{p}{6})$.

(iii) If $D \equiv -1 \pmod{3}$ then $p' = p + 2(D + 1)$ iff $N(\frac{p'}{6}) = \frac{D+1}{3} + N(\frac{p}{6})$.

Proof. (i) $\frac{p' \mp 1}{6} = \frac{D}{3} + \frac{p \mp 1}{6}$ is equivalent to $p' = p + 2D$. (ii) $\frac{p' \mp 1}{6} = \frac{D-1}{3} + \frac{p \mp 1}{6}$ is equivalent to $p' = p + 2(D - 1)$. (iii) $\frac{p' \mp 1}{6} = \frac{D+1}{3} + \frac{p \mp 1}{6}$ is equivalent to $p' = p + 2(D + 1)$. \diamond

It is straightforward to relax the constraint $p' \equiv p \pmod{6}$ to include $p' \equiv p \pm 2 \pmod{6}$.

We now consider systematically common (or double) non-ranks of pairs of primes. We start with ordinary twin primes.

Theorem 3.4. Let $p' > p \geq 5$ and $N(\frac{p'}{6}) = N(\frac{p}{6})$. Then (i)

$$pp'n \pm 3DN(\frac{p}{6}) = p'pn \pm 3DN(\frac{p'}{6}) \quad (19)$$

are two common non-rank progressions of p and p' . (ii) If r, r' solve

$$\begin{aligned} (r' - r \pm D)p &= 2r \pm D, \quad p \equiv 1 \pmod{6} \\ (r' - r \pm D)p &= 2r \mp D, \quad p \equiv -1 \pmod{6}, \end{aligned} \quad (20)$$

then

$$p(p'n + r') \pm 3DN(\frac{p}{6}) = p'(pn + r) \mp 3DN(\frac{p'}{6}) \quad (21)$$

are the other two common non-rank progressions of p and p' .

Proof. By Lemma 3.2, $p' = p + 2$ and p are ordinary twin primes and Eq. (19) is valid obviously, with the lhs a non-rank to p and the rhs a non-rank to p' . (ii) If $2r = \mp D + \lambda p$, $r' = r + \lambda \mp D$ for odd λ so that $-p < 2r < p$, $-p' < 2r' < p'$, solving Eq. (20) for $p \equiv 1 \pmod{6}$, then Eq. (21) is verified to be equivalent to Eq. (20), its lhs being a non-rank to p and rhs a non-rank to p' . For $p \equiv -1 \pmod{6}$ in Eqs. (20),(21) the cases are treated similarly. \diamond

Theorem 3.5. Let $p' > p \geq 5$ be primes with $(p, D) = 1 = (p', D)$. (i) If $p' \equiv p \pmod{6}$, then $p' = p + 6l$, $l \geq 1$, $N(\frac{p'}{6}) = N(\frac{p}{6}) + l$ and two common non-rank progressions of p', p are

$$p[p'n + r'] \pm 3DN(\frac{p}{6}) = p'[pn + r] \pm 3DN(\frac{p'}{6}) \quad (22)$$

provided r, r' solve

$$(r' - r)p = 3l(2r \pm D). \quad (23)$$

The solution of Eq. (23), $2r = \mp D + p\lambda_{\pm}$, $-p < 2r < p$ for odd λ_{\pm} with $r' = r + \lambda_{\pm}$, $-p' < 2r' < p'$ on the lhs of Eq. (22) yields a non-rank to p and, on the rhs, a non-rank to p' .

If r, r' solve

$$\begin{aligned} (r' - r \pm D)p &= 6lr \mp D(3l - 1), \quad p \equiv 1 \pmod{6} \\ (r' - r \pm D)p &= 6lr \mp D(3l + 1), \quad p \equiv -1 \pmod{6}, \end{aligned} \quad (24)$$

then two more common non-rank progressions are

$$p[p'n + r'] \pm 3DN(\frac{p}{6}) = p'[pn + r] \mp 3DN(\frac{p'}{6}). \quad (25)$$

(ii) If $p' \equiv 1 \pmod{6}$, $p \equiv -1 \pmod{6}$ then $p' = p + 6l + 2$, $l \geq 0$, $N(\frac{p'}{6}) = N(\frac{p}{6}) + l$, and two common non-rank progressions of p', p are

$$p[p'n + r'] \pm 3DN(\frac{p}{6}) = p'[pn + r] \pm 3DN(\frac{p'}{6}) \quad (26)$$

provided

$$(r' - r)p = 2r(3l + 1) \pm 3lD. \quad (27)$$

If $l = 0$ then $r' = r = 0$; see Theor. 3.4. For $l \geq 1$, $2r(3l + 1) = \mp 3lD + p\lambda$, $r' = r + \lambda$ solve Eq. (27). There is a unique pair r', r with $-p < 2r < p$, $-p' < 2r' < p'$.

If r, r' solve

$$(r' - r \pm D)p = 3l(2r \mp D) + 2r, \quad (28)$$

then two more common non-rank progressions are

$$p[p'n + r'] \pm 3DN(\frac{p}{6}) = p'[pn + r] \mp 3DN(\frac{p'}{6}). \quad (29)$$

For appropriate λ , the solution $r' = r + \lambda \mp D$, $3l(2r \mp D) + 2r = p\lambda$ is unique.

(iii) If $p' \equiv -1 \pmod{6}$, $p \equiv 1 \pmod{6}$ then $p' = p + 6l - 2$, $l \geq 1$, $N(\frac{p'}{6}) = N(\frac{p}{6}) + l$, and two common non-rank progressions of p', p are

$$p[p'n + r'] \pm 3DN(\frac{p}{6}) = p'[pn + r] \pm 3DN(\frac{p'}{6}) \quad (30)$$

provided

$$(r' - r)p = 2r(3l - 1) \pm 3lD. \quad (31)$$

Again, for appropriate λ the solution $r' = r + \lambda$, $2r(3l - 1) = p\lambda \mp 3lD$ is unique.

If r, r' solve

$$(r' - r \pm D)p = 2r(3l - 1) \mp 3lD, \quad (32)$$

then two more common non-rank progressions are

$$p[p'n + r'] \pm 3DN\left(\frac{p}{6}\right) = p'[pn + r] \mp 3DN\left(\frac{p'}{6}\right). \quad (33)$$

The solution $2r(3l-1) = \pm 3lD + p\lambda$, $r' = r + \lambda \mp D$ is unique for appropriate λ .

Note that there are $4 = 2^{\nu(pp')}$ arithmetic progressions of common or double non-ranks to the primes p', p in all cases.

Proof. By substituting $p', N(p'/6)$ in terms of $p, N(p/6)$ and l , respectively, it is readily verified that Eqs. (22), (23) are equivalent, as are (24), (25), and (28), (29), and (30), (31), and (32), (33). As in (i) there is a unique solution (r, r') in all other cases as well. \diamond

Theorem 3.6. (Triple non-ranks) *Let $5 \leq p < p' < p''$ (or $5 \leq p < p'' < p'$, or $5 \leq p'' < p < p'$) be different odd primes such that $(p, D) = 1 = (p', D) = (p'', D)$. Then each case in Theor. 3.5 of four double non-ranks leads to $8 = 2^{\nu(pp'p'')}$ triple non-ranks of p, p', p'' . At two non-ranks per prime, there are at most 2^3 triple non-ranks.*

Proof. It is based on Theor. 3.5 and similar for all its cases. Let's take (i) and substitute $n \rightarrow p''n + \nu$, $0 \leq \nu < p''$ in Eq. (22) which, upon dropping the term $p''p'pn$, yields on the lhs

$$pp'\nu + pr' - 3DN\left(\frac{p}{6}\right) = p''\mu \pm 3DN\left(\frac{p''}{6}\right). \quad (34)$$

Since $(pp', p'') = 1$ there is a unique residue ν modulo p'' so that the lhs of Eq. (34) is $\equiv \pm 3DN\left(\frac{p''}{6}\right) \pmod{p''}$, and this determines μ . As each sign case leads to such a triple non-rank solution, it is clear that there are 2^3 non-ranks to p, p', p'' . \diamond

Theorem 3.7. (Multiple non-ranks) *Let $5 \leq p_1 < \dots < p_m$ be m different primes with $(p_i, D) = 1$. Then there are 2^m arithmetic progressions of m -fold non-ranks to the primes p_1, \dots, p_m .*

Proof. This is proved by induction on m . Theors. 3.5 and 3.6 are the $m = 2, 3$ cases. If Theor. 3.7 is true for m then for any case $5 \leq p_{m+1} < p_1 < \dots < p_m$, or \dots , $5 \leq p_1 < \dots < p_{m+1}$, we substitute in an m -fold non-rank equation $n \rightarrow p_{m+1}n + \nu$ as in the proof of Theor. 3.6, again dropping the $n \prod_{i=1}^{m+1} p_i$ term. Then we get

$$\begin{aligned} & p_1(p_2(\dots(p_m\nu + r_m) + \dots + r_2) + 3DN\left(\frac{p_1}{6}\right) \\ &= p_{m+1}\mu \pm 3DN\left(\frac{p_{m+1}}{6}\right) \end{aligned} \quad (35)$$

with a unique residue $\nu \pmod{p_{m+1}}$ so that the lhs of Eq. (35) becomes $\equiv 3DN(\frac{p_{m+1}}{6}) \pmod{p_{m+1}}$, which then determines μ . In case the lhs of Eq. (35) has $p_1(\dots) - 3DN(p_1/6)$ the argument is the same. This yields an $(m+1)$ -fold non-rank progression since each sign in Eq. (35) gives a solution. Hence there are 2^{m+1} such non-ranks. At two non-ranks per prime, there are at most 2^{m+1} non-rank progressions. \diamond

4 Counting Non-Ranks

If we subtract for case (i) in Theor. 3.5, say, the four common non-rank progressions corresponding to the solutions $-p_i < r_i < p_i, \dots$, this leaves in $\mathcal{A}_p^- = \{p'n \pm 3D\frac{p'+1}{6}\}$ the following progressions $p'pn \pm 3D\frac{p'+1}{6}, \dots, p'[np + r_1] + 3D\frac{p'+1}{6}, \dots, p'[np + r_2] - 3D\frac{p'+1}{6}, \dots, p'[np + r_3] + 3D\frac{p'+1}{6}, \dots, p'[np + r_4] - 3D\frac{p'+1}{6}, \dots, p'np \pm 3D\frac{p'+1}{6}$.

We summarize this as follows.

Lemma 4.1. *$p' > p \geq 5$ be prime such that $(p, D) = 1 = (p', D)$. Removing the nontrivial common non-ranks of p', p from the set of all non-ranks of p' leaves arithmetic progressions of the form $p'np + l$; $n \geq 0$, where $l > 0$ are given nonnegative integers.*

Proposition 4.2. *Let $p \geq p' \geq 5$ be prime with $(p, D) = 1$. Then the set of non-ranks to parent prime p , \mathcal{A}_p , is made up of arithmetic progressions $\bar{L}(p)n + a$, $n \geq 0$ with $\bar{L}(p) = \prod_{5 \leq p' \leq p, (p', D)=1} p'$ and $a > 0$ given integers.*

Proof. Let $p = 6m \pm 1$. We start from the set $\mathcal{A}_p^\pm = \{pn \pm 3DN(\frac{p}{6})\}$. Removing the non-ranks common to p and 5 by Theor. 3.5 leaves arithmetic progressions of the form $5pn + l$, $n \geq 0$ where $l > 0$ are given integers provided $5 \nmid D$. Continuing this process to the largest prime $p' < p$ leaves in \mathcal{A}_p arithmetic progressions of the form $\bar{L}(p)n + a$, $n \geq 0$ with $L(p) = \prod_{5 \leq p' \leq p, (p', D)=1} p'$ and $a > 0$ a sequence of given integers independent of n . \diamond

Proposition 4.3. *Let $p \geq p' \geq 5$ be primes such that $(p, D) = 1 = (p', D)$ and $G(p)$ the number of nontrivial non-ranks $\bar{L}(p)n + a \in \mathcal{A}_p$ over one period $\bar{L}(p)$ corresponding to arithmetic progressions $\bar{L}(p)n + a \in \mathcal{A}_p$. Then $G(p) = 2 \prod_{5 \leq p' < p, (p', D)=1} (p' - 2)$.*

Note that $G(p) < \bar{L}(p)$ both increase monotonically as $p \rightarrow \infty$.

Proof. In order to determine $G(p)$ we have to eliminate all non-ranks of primes $5 \leq p' < p$ from \mathcal{A}_p . As in Theor. 3.5 we start by subtracting the fraction $2/5$ of non-ranks to $p' = 5$ from the interval $1 \leq a \leq \bar{L}(p)$, then $2/7$ for $p' = 7$ and so on for all $p' < p$. The factor of 2 is due to the

symmetry of non-ranks around each multiple of p' according to Lemma 2.3. This leaves $p \prod_{5 \leq p' < p, (p', D)=1} (p' - 2)$ numbers a . The fraction $2/p$ of these are the non-ranks to parent prime p . \diamond

Prop. 4.3 implies that the fraction of non-ranks related to a prime p in the interval occupied by \mathcal{A}_p ,

$$q(p) = \frac{G(p)}{\bar{L}(p)} = \frac{2}{p} \prod_{5 \leq p' < p, (p', D)=1} \frac{p' - 2}{p'}, \quad (36)$$

where p' is prime, decreases monotonically as p goes up.

Definition 4.4. Let $p \geq p' \geq 5$ be prime such that $(p, D) = 1 = (p', D)$. The supergroup $\mathcal{S}_p = \bigcup_{5 \leq p' \leq p, (p', D)=1} \mathcal{A}_{p'}$ contains the sets of non-ranks corresponding to arithmetic non-rank progressions $a + \bar{L}(p')n$ of all $\mathcal{A}_{p'}$, $5 \leq p' \leq p$; $(p', D) = 1 = (p, D)$.

Thus, each supergroup \mathcal{S}_p contains nested sets of non-ranks related to primes $5 \leq p' \leq p$, $(p', D) = 1 = (p, D)$

Let us now count prime numbers from $p_1 = 5$ on provided $5 \nmid D$, omitting prime divisors of D along with 2 and 3.

Proposition 4.5. Let $p_j \geq 5$ be the j th prime. (i) Then the number of nontrivial non-ranks $a \in \mathcal{A}_{p_i}$ corresponding to arithmetic progressions related to a prime $p_i < p_j$, $(p_i, D) = 1 = (p_j, D)$

$$G(p_i) = \frac{\bar{L}(p_j)}{\bar{L}(p_i)} G(p_j) = \frac{2\bar{L}(p_j)}{p_i} \prod_{5 \leq p < p_i; (p, D)=1=(p_i, D)} \frac{p - 2}{p} = q(p_i) \bar{L}(p_j), \quad (37)$$

where p is prime, monotonically decreases as p_i goes up. (ii) The number of nontrivial non-ranks in a supergroup \mathcal{S}_{p_j} over one period $\bar{L}(p_j)$ is

$$S(p_j) = \bar{L}(p_j) \sum_{5 \leq p \leq p_j; (p, D)=1=(p_j, D)} q(p) = \bar{L}(p_j) \left(1 - \prod_{5 \leq p \leq p_j; (p, D)=1=(p_j, D)} \frac{p - 2}{p} \right). \quad (38)$$

(iii) The fraction of non-ranks of their arithmetic progressions in the (first) interval $[1, \bar{L}(p_j)]$ occupied by the supergroup \mathcal{S}_{p_j} ,

$$Q(p_j) = \frac{S(p_j)}{\bar{L}(p_j)} = \sum_{5 \leq p \leq p_j; (p, D)=1=(p_j, D)} q(p) = 1 - \prod_{5 \leq p \leq p_j; (p, D)=1=(p_j, D)} \frac{p - 2}{p}, \quad (39)$$

increases monotonically as p_j goes up.

Proof. (i) follows from Prop. 4.3 and Eq. (36). (ii) and (iii) are equivalent and are proved by induction as follows, using Def. 4.4 in conjunction with Eq. (36).

From Eq. (36) we get $q_1 = 2/p_1$ which is the case $j = 1$, $p_j = 5$ of Eq. (39). Assuming Eq. (39) for p_j , we add q_{j+1} of Eq. (36) and obtain

$$\begin{aligned} \sum_{i=1}^{j+1} q(p_i) &= 1 - \prod_{i=1}^j \frac{p_i - 2}{p_i} + \frac{2}{p_{j+1}} \prod_{i=1}^j \frac{p_i - 2}{p_i} \\ &= 1 - \prod_{i=1}^{j+1} \frac{p_i - 2}{p_i}. \end{aligned} \quad (40)$$

The extra factor $0 < (p_{j+1} - 2)/p_{j+1} < 1$ shows that $q(p_j), x(p_j)$ in Eq. (42) decrease monotonically as $p_j \rightarrow p_{j+1}$ while $Q(p_j)$ increases as $j \rightarrow \infty$. \diamond

Definition 4.6. Since $\bar{L}(p) > S(p)$, there is a set \mathcal{R}_p of *remnants* $r \in [1, \bar{L}(p)]$ such that $r \notin \mathcal{S}_p, (r, D) = 1$.

Lemma 4.7. (i) *The number $R(p_j)$ of remnants in a supergroup, \mathcal{S}_{p_j} , with $(p_j, D) = 1, p_j$ prime is*

$$\begin{aligned} R(p_j) &= \bar{L}(p_j) - S(p_j) = \bar{L}(p_j)(1 - Q(p_j)) = \prod_{5 \leq p \leq p_j; (p, D)=1=(p_j, D)} (p - 2) \\ &= \frac{1}{2} G(p_{j+1}). \end{aligned} \quad (41)$$

(ii) *The fraction of such remnants in \mathcal{S}_{p_j} ,*

$$x(p_j) = \frac{R(p_j)}{\bar{L}(p_j)} = 1 - Q(p_j) = \prod_{5 \leq p \leq p_j; (p, D)=1=(p_j, D)} \frac{p - 2}{p}, \quad (42)$$

where p is prime, decreases monotonically as $p_j \rightarrow \infty$.

Proof. (i) follows from Def. 4.6 in conjunction with Eq. (38) and (ii) from Eq. (41). Equation (41) follows from Eq. (39).

5 Remnants and Twin Ranks

When all primes $5 \leq p \leq p_j, (p_j, D) = 1$ and appropriate nonnegative integers n are used in Lemma 2.3 one will find all non-ranks $k < M(j + 1) \equiv (p_{j+1}^2 - D^2)/2$. By subtracting these non-ranks from the set of positive integers $N \leq M(j + 1)$ all and only twin-D-I ranks $t < M(j + 1)$ are left among

the remnants provided trivial non-ranks are also eliminated. If a non-rank k is left then $2k \pm D$ must have prime divisors that are $> p_j$ according to Lemma 2.3, which is impossible.

Definition 5.1. Let $p_j, p_{j+1} \geq 5$ be prime such that $(p_j, D) = 1 = (p_{j+1}, D)$. Then all $t < M(j+1) = (p_{j+1}^2 - D^2)/2$ in a remnant \mathcal{R}_{p_j} of a supergroup \mathcal{S}_{p_j} are twin-D-I ranks. These twin ranks are called *front twin ranks*.

Twin ranks are located among the remnants \mathcal{R}_p for any prime $p \geq 5, (p, D) = 1$. Our goal is to determine the number of twin-D-I ranks.

Theorem 5.2. Let R_0 be the number of remnants of the supergroup \mathcal{S}_{p_j} , where p_j is the j th prime number with $(p_j, D) = 1, p_j > p, \forall p|D$ and $M(j+1) = (p_{j+1}^2 - D^2)/2$. Then the number $R = \pi_2(2\bar{L}(p_j) + D)/2$ of twin-D-I ranks within the remnants of the supergroup \mathcal{S}_{p_j} is given by

$$R \prod_{p|D} (1 - \frac{1}{p})^{-1} = R_0 + \sum_{(n,D)=1, p_j > n} \mu(n) 2^{\nu(n)} \left[\frac{\bar{L}(p_j) - M(j+1)}{n} \right]. \quad (43)$$

Here $\bar{L}(p_j) = \prod_{5 \leq p \leq p_j, (p,D)=1} p$, $R_0 = \prod_{5 \leq p \leq p_j, (p,D)=1} (p-2)$ with p prime, and n runs through all products of primes $p_j < p \leq (2\bar{L}(p_j) + 1)/D$ relatively prime to D . The upper limit $(2\bar{L}(p_j) + 1)/D$ comes about because $3DN(p/6)$ is the lowest possible non-rank of a prime number p according to Lemma 2.2.

The argument of the twin-prime counting function π_2 is $2\bar{L}(p_j) + D$ because, if $\bar{L}(p_j)$ is the last twin-D-I rank of the interval $[1, \bar{L}(p_j)]$, then $2\bar{L}(p_j) \pm D$ are the corresponding twin-D-I primes.

Proof. According to Prop. 4.5 the supergroup \mathcal{S}_{p_j} has $S(p_j) = \bar{L}(p_j) \cdot (1 - \prod_{5 \leq p \leq p_j} \frac{p-2}{p})$ non-ranks. Subtracting these from the interval $[1, \bar{L}(p_j)]$ that the supergroup occupies gives $R_0 = \prod_{5 \leq p \leq p_j, (p,D)=1} (p-2)$ for the number of remnants which include twin-D-I ranks and non-ranks to primes $p_j < p \leq (2\bar{L}(p_j) + 1)/D$. The latter are

$$M(j+1) < pn \pm 3DN(\frac{p}{6}) \leq \bar{L}(p_j), \quad M(j+1) = (p_{j+1}^2 - D^2)/2, \quad (44)$$

or

$$0 < n \leq \frac{\bar{L}(p_j) - M(j+1)}{p}, \quad (45)$$

which have to be subtracted from the remnants to leave just twin-D-I ranks. Correcting for double counting of common non-ranks to two primes using

Theor. 3.5, of triple non-ranks using Theor. 3.6 and multiple non-ranks using Theor. 3.7 and eliminating trivial non-ranks to prime divisors of D , which leads to the factor on the lhs of Eq. (43), we obtain

$$R \prod_{p|D} \left(1 - \frac{1}{p}\right)^{-1} = R_0 - 2 \sum_{p_j < p \leq (2\bar{L}(p_j)+1)/D, (p,D)=1} \left[\frac{\bar{L}(p_j) - M(j+1)}{p} \right] \\ + 4 \sum_{p_j < p < p' \leq (2\bar{L}(p_j)+1)/D, (p,D)=1} \left[\frac{\bar{L}(p_j) - M(j+1)}{pp'} \right] \mp \dots, \quad (46)$$

where $[x]$ is the integer part of x as usual. Equation (46) is equivalent to Eq. (43). \diamond

Definition 5.3. We split $R = R_M + R_E$ into its main and error terms

$$R_M \prod_{p|D} \left(1 - \frac{1}{p}\right)^{-1} = R_0 - 2 \sum_{p_j < p \leq (2\bar{L}(p_j)+1)/D, (p,D)=1} \frac{\bar{L}(p_j) - M(j+1)}{p} \\ + 4 \sum_{p_j < p < p' \leq (2\bar{L}(p_j)+1)/D, (p,D)=1} \frac{\bar{L}(p_j) - M(j+1)}{pp'} \mp \dots, \quad (47)$$

$$R_E = 2 \sum_{p_j < p \leq (2\bar{L}(p_j)+1)/D, (p,D)=1} \left\{ \prod_{q|D} \left(1 - \frac{1}{q}\right) \frac{\bar{L}(p_j) - M(j+1)}{p} \right\} \quad (48)$$

$$- 4 \sum_{p_j < p < p' \leq (2\bar{L}(p_j)+1)/D, (p,D)=1} \left\{ \prod_{q|D} \left(1 - \frac{1}{q}\right) \frac{\bar{L}(p_j) - M(j+1)}{pp'} \right\} \mp \dots \quad (49)$$

using the usual decomposition $[x] = x - \{x\}$.

Theorem 5.4. The main term R_M satisfies

$$R_M \prod_{p|D} \left(1 - \frac{1}{p}\right)^{-1} = \bar{L}(p_j) \prod_{5 \leq p \leq (2\bar{L}(p_j)+1)/D, (p,D)=1} \left(1 - \frac{2}{p}\right) \\ + M(j+1) \left[1 - \prod_{p_j < p \leq (2\bar{L}(p_j)+1)/D, (p,D)=1} \left(1 - \frac{2}{p}\right)\right]. \quad (50)$$

Proof. Expanding the product

$$\prod_{5 \leq p \leq p_j, (p,D)=1} \left(1 - \frac{2}{p}\right) \quad (51)$$

and combining corresponding sums in Eq. (49)

$$\begin{aligned}
& - \sum_{5 \leq p \leq p_j, (p, D)=1=(p_j, D)} \frac{1}{p} - \sum_{p_j < p \leq (2\bar{L}(p_j)+1)/D, (p, D)=1} \frac{1}{p} \\
& = - \sum_{5 \leq p \leq (2\bar{L}(p_j)+1)/D, (p, D)=1} \frac{1}{p}, \dots
\end{aligned} \tag{52}$$

shifts the upper limit of the primes in the product $\prod_p (1 - 2/p)$ from p_j to $(2\bar{L}(p_j) + 1)/D$ so that we obtain Eq. (50). The considerable cancellations involved collapse R_0 to the expected magnitude in R_M . \diamond

Theorem 5.5. *The main term R_M obeys the asymptotic law*

$$R_M \sim \frac{\prod_{p|D} (1 - \frac{1}{p}) 6c_2 e^{-2\gamma} 2\bar{L}(p_j)}{\prod_{p|D} (1 - \frac{2}{p}) \log^2(2\bar{L}(p_j) + 1)/D}, \tag{53}$$

for $p_j \rightarrow \infty$, where $\bar{L}(p_j) = \prod_{5 \leq p \leq p_j, (p, D)=1} p$.

Proof. This follows from Theorem 5.4 as in the proof of Theor. 5.8 in Ref. [5]. \diamond

6 Summary and Discussion

The twin prime sieves constructed here are genuine asymptotic pair sieves that work only for prime twins at odd half-distance $D \geq 3$.

Accurate counting of non-rank sets require the infinite, but sparse set of odd ‘primorials’ $\{\bar{L}(p_j) = \prod_{3 < p \leq p_j, (p, D)=1} p\}$. The twin primes are not directly sieved, rather twin-D-I ranks m are with $2m \pm D$ both prime. All other numbers are non-ranks. Primes serve to organize and classify (nontrivial) non-ranks in arithmetic progressions with equal distances (periods) that are primes or products of them.

The coefficients of the asymptotic twin-D-I prime distributions depend on D . They are ≈ 5 for sexy primes with $D = 3$, ≈ 3.3 for $D = 5$, and ≈ 2.49 for (sufficiently) large $D = \text{prime}$ for which $\prod_{p|D} (1 - \frac{1}{p})(1 - \frac{2}{p})^{-1} = 1 + \varepsilon$, for some $\varepsilon > 0$. This is about a factor 3 larger than for ordinary twins and, remarkably, reflects the different abundances allowed by their progressions $2m \pm D$ and $2(3m) \pm 1$ in class I. If the distance $2D = 2 \prod_{3 \leq p \leq p_e} p$, then the coefficient grows as $\log p_e$.

Thus, pair sieves as a resolution of the parity problem for prime twins in class I allow replacing the need for a lower bound on the number of twin-D-I ranks R (or $\pi_2/2$) by an upper bound for the remainder R_E (that must be lower than R_M).

References

- [1] H. Halberstam and H. E. Richert, *Sieve Methods*, Acad. Press, New York, 1974; Dover, New York (2011).
- [2] M. Ram Murty, *Problems in Analytic Number Theory*, Springer, New York (2001).
- [3] H. Riesel, *Prime Numbers and Computer Methods for Factorization*, 2nd ed., Birkhäuser, Boston (1994).
- [4] J. Friedlander and H. Iwaniec, *Opera Cribro*, Amer. Math. Soc. Colloq. Publ. **59** (2010), Prov. RI.
- [5] A. Dinculescu and H. J. Weber, *Twin Prime Sieve*, [www.arXiv.org/1203.5240](http://www.arXiv.org/abs/1203.5240)
- [6] H. J. Weber, *A Sieve for Cousin Primes*, [www.arXiv.org/1204.3795](http://www.arXiv.org/abs/1204.3795)
- [7] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 5th ed. (1988).
- [8] H. J. Weber, *Regularities of Prime Number Twins, Triplets and Multiplets*, *Global J. Pure Applied Math.* **8** (2012), www.adsabs.harvard.edu/abs/2011arXiv1103.0447W.
- [9] H. J. Weber, *Exceptional Prime Number Twins, Triplets and Multiplets*, [www.arXiv.org:1102.3075](http://www.arXiv.org/abs/1102.3075).